ON LOW-FREQUENCY OSCILLATIONS OF A PLATE ON ELASTIC HALF-SPACE

(O NIZKOCHASTOTNYKH KOLEBANIIAKH PLASTINY NA UPRUGOM POLUPROSTRANSTVE)

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Through the use of exact solutions of problems in the dynamic theory of elasticity for stratified systems, the special properties of wave propagation in the low-frequency range have been clarified. The study of the displacement field in a homogeneous thin elastic layer has opened up the possibility of justifying and redefining more exactly the well-known classical equations of oscillations of thin plates [1]. Likewise, as a result of this approach the equations of oscillations of multiple-layered plates were written down [2] and elastic-fluid systems consisting of half-spaces divided by a thin layer were investigated [3,4]. This method is applied below to the study of low-frequency oscillations of an elastic layer which is in a nonrigid contact with the underlying elastic half-space. Let us note that by low-frequency oscillations we mean those whose frequency f satisfies the condition $f \ll v_s/2h$ (v_s is the velocity of the transverse waves and h is the thickness of the layer). This condition is equivalent to the requirement of a small thickness of the layer as compared to the wave length of oscillations.

1. An elastic layer (0) ($0 < z < h$) situated on the elastic half-space (1) ($z > h$) is given in a cylindrical coordinate system. We designate by a_0^{-1}, a_1^{-1}, b_0^{-1} and b_1^{-1} the velocities of longitudinal and transverse waves in the media (0) and (1) respectively; by μ_0, μ_1, ρ_0 and ρ_1 the moduli of shear and density; and by φ_0, ψ_0, φ_1 and ψ_1 the longitudinal and transverse potentials of the displacement fields which satisfy the wave equations

$$\frac{\partial^2\varphi_\nu}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_\nu}{\partial r} + \frac{\partial^2\varphi_\nu}{\partial z^2} = a_\nu^2\frac{\partial^2\varphi_\nu}{\partial t^2}$$

$$\frac{\partial^2\psi_\nu}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_\nu}{\partial r}-\frac{\psi_\nu}{r}+\frac{\partial^2\psi_\nu}{\partial z^2}=b_\nu^2\frac{\partial^2\psi_\nu}{\partial t^2}\qquad(\nu=0,1)\tag{1.1}$$

We will assume that on the boundary $z = h$ nonrigid contact takes place: the normal components of displacement w and stress t_{zz} are continuous, the shear stress $t_{rz} = 0$. These conditions lead to equations

$$2\frac{\partial^2\varphi_0}{\partial r\partial z}+b_0^2\frac{\partial^2\psi_0}{\partial t^2}-2\frac{\partial^2\psi_0}{\partial z^2}=0,\qquad 2\frac{\partial^2\varphi_1}{\partial r\partial z}+b_1\frac{\partial^2\psi_1}{\partial t^2}-2\frac{\partial^2\psi_1}{\partial z^2}=0$$

$$\mu_0\left[(b_0^2-2a_0^2)\frac{\partial^2\varphi_0}{\partial t^2}+2\frac{\partial^2\varphi_0}{\partial z^2}+2\frac{\partial^2\psi_0}{\partial r\partial z}+\frac{2}{r}\frac{\partial\psi_0}{\partial z}\right]=$$
$$=\mu_1\left[(b_1^2-2a_1^2)\frac{\partial^2\varphi_1}{\partial t^2}+2\frac{\partial^2\varphi_1}{\partial z^2}+2\frac{\partial^2\psi_1}{\partial r\partial z}+\frac{2}{r}\frac{\partial\psi_1}{\partial z}\right]\tag{1.2}$$
$$\frac{\partial\varphi_0}{\partial z}+\frac{\partial\psi_0}{\partial r}+\frac{\psi_0}{r}=\frac{\partial\varphi_1}{\partial z}+\frac{\partial\psi_1}{\partial r}+\frac{\psi_1}{r}$$

relating potentials φ_0, ψ_0, φ_1, ψ_1 for $z = h$.

For $t < 0$ there is no disturbance in the medium, but at $t = 0$ at point $z = 0$, $r = 0$ a source in the form of a normal force begins to act

$$t_{rz}^\circ=0,\qquad t_{zz}^\circ=\int_0^\infty\frac{k^2J_0(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}A(k,\eta)\exp\left(\frac{kt\eta}{b_0}\right)d\eta\tag{1.3}$$

The function $A(k, \eta)$ is determined by the functional dependence of the source upon time. In all further investigations the form of the function $A(k, \eta)$ proves to be immaterial.

Thus the determination of the displacement potentials φ_0, φ_1, ψ_0 and ψ_1 is reduced to the solution of equations (1.1) with zero initial and boundary conditions (1.2) and (1.3).

2. We look for the solution of the problem in the form

$$\varphi_0=\int_0^\infty\frac{J_0(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}[X_0^+e^{kz\alpha_0}+X_0^-e^{-kz\alpha_0}]\exp\left(\frac{kt\eta}{b_0}\right)d\eta$$
$$\psi_0=\int_0^\infty\frac{J_1(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}[Y_0^+e^{kz\beta_0}+Y_0^-e^{-kz\beta_0}]\exp\left(\frac{kt\eta}{b_0}\right)d\eta$$
$$\varphi_1=\int_0^\infty\frac{J_0(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}X_1\exp\left\{k\left[\frac{t\eta}{b_0}-(z-h)\,\alpha_1\right]\right\}d\eta\tag{2.1}$$
$$\psi_1=\int_0^\infty\frac{J_1(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}Y_1\exp\left\{k\left[\frac{t\eta}{b_0}-(z-h)\,\beta_1\right]\right\}d\eta$$

The following designations were used in formulas (2.1):

$$\beta_0 = \sqrt{1+\eta^2}, \quad \beta_1 = \sqrt{1+\delta_1^2\eta^2}, \quad \alpha_0 = \sqrt{1+\gamma_0^2\eta^2}$$
$$\alpha_1 = \sqrt{1+\gamma_1^2\eta^2}, \ \delta_1 = b_1 b_0^{-1}, \quad \gamma_0 = a_0 b_0^{-1}, \quad \gamma_1 = a_1 b_0^{-1}$$

In order to have single-valued roots β_0, β_1, α_0 and α_1 the branch cuts were drawn from the branch points $\pm\, i$, $\pm\, i\delta_1^{-1}$ $\pm\, i\gamma_0^{-1}$ and $\pm\, i\gamma_1^{-1}$ into the left half-plane and the main branches were fixed by the conditions $\arg \alpha_i = \arg \beta_i = 0$ for $\eta > 0$. Let us note that the solutions in form (2.1) satisfy equations (1.1) and zero initial conditions. Substituting (2.1) into boundary conditions (1.2) to (1.3) and solving the system of algebraic equations, we obtain the expressions for the functions $X_0^{\pm}$, $Y_0^{\pm}$, X_1 and Y_1.

In the analysis of the solutions let us pay particular attention to the components q_ν and w_ν of the displacement vector $u_\nu = q_\nu \mathbf{r}_1 + w_\nu \mathbf{k}_1$ in the νth medium. These components are related to the potentials φ_ν and ψ_ν by the expressions

$$q_\nu = \frac{\partial\varphi_\nu}{\partial r} - \frac{\partial\psi_\nu}{\partial z}, \qquad w_\nu = \frac{\partial\varphi_\nu}{\partial z} + \frac{\partial\psi_\nu}{\partial r} + \frac{\psi_\nu}{r} \tag{2.2}$$

Based on solutions (2.1) and expressions (2.2) for the displacements q_0 and w_0 on the daylight surface and displacements q_1 and w_1 in the half-space, we have

$$q_0 = \frac{1}{\mu_0}\int_0^\infty \frac{kJ_1(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Q_0(k,\eta)\,A(k,\eta)\exp\left(\frac{kt\eta}{b_0}\right)d\eta$$
$$w_0 = \frac{1}{\mu_0}\int_0^\infty \frac{kJ_0(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} W_0(k,\eta)\,A(k,\eta)\exp\left(\frac{kt\eta}{b_0}\right)d\eta \tag{2.3}$$
$$q_1 = \frac{1}{\mu_0}\int_0^\infty \frac{kJ_1(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} [Q_1^{(1)}e^{k(h-z)\alpha_1} + Q_2^{(1)}e^{k(h-z)\beta_1}]\exp\left(\frac{kt\eta}{b_0}\right)d\eta A(k,\eta)$$
$$w_1 = \frac{1}{\mu_0}\int_0^\infty \frac{kJ_0(kr)\,dk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} [W_1^{(1)}e^{k(h-z)\alpha_1} + W_2^{(1)}e^{k(h-z)\beta_1}]\exp\left(\frac{kt\eta}{b_0}\right)d\eta\, A(k,\eta)$$

The designations used in equalities (2.3) were

$$Q_0 = -\Delta_1\Delta_0^{-1}, \quad W_0 = \Delta_2\Delta_0^{-1}, \quad Q_1^{(1)} = -\Delta_3\Delta_0^{-1}, \quad Q_2^{(1)} = -\Delta_4\Delta_0^{-1} \tag{2.4}$$
$$W_1^{(1)} = -\Delta_5\Delta_0^{-1}, \quad W_2^{(1)} = -\Delta_6\Delta_0^{-1}$$
$$\Delta_0 = -R_1\alpha_0\,[g_0^2 \sinh kh\beta_0 \cosh kh\alpha_0 - 4\alpha_0\beta_0 \sinh kh\alpha_0 \cosh kh\beta_0] +$$
$$+ \sigma_{01}\delta_1^2\alpha_1\,[8\alpha_0\beta_0 g_0^2(\cosh kh\alpha_0 \cosh kh\beta_0 - 1) - (g_0^4 + 16\alpha_0^2\beta_0^2)\sinh kh\alpha_0 \sinh kh\beta_0]$$

$$\Delta_1 = -R_1\alpha_0 (g_0^2 \sinh kh\beta_0 \cosh kh\alpha_0 - 2\alpha_0\beta_0 \sinh kh\alpha_0 \cosh kh\beta_0) +$$
$$+ 2\sigma_{01}\delta_1^2\alpha_0\beta_0\alpha_1 (g_0^2 + 2) g_0 (\cosh kh\alpha_0 \cosh kh\beta_0 - 1) - \sigma_{01}\delta_1^2\alpha_1 (g_0^3 + 8\alpha_0^2\beta_0^2) \sinh kh\alpha_0 \sinh kh\beta_0$$
$$\Delta_2 = \alpha_0\eta^2 \{\alpha_0 R_1 \sinh kh\alpha_0 \sinh kh\beta_0 + \sigma_{01}\delta_1^2\alpha_1 [g_0^2 \sinh kh\beta_0 \cosh kh\alpha_0 - 4\alpha_0\beta_0 \sinh kh\alpha_0 \cosh kh\beta_0]\}$$
$$\Delta_3 = \sigma_{01}\alpha_0 g_1 (g_0^2 \sinh kh\beta_0 - 4\alpha_0\beta_0 \sinh kh\alpha_0)$$
$$g_1\Delta_4 = -2\alpha_1\beta_0\Delta_3, \qquad \Delta_5 = \Delta_0\Delta_3, \qquad \beta_1\Delta_6 = \Delta_4, \qquad R_1 = g_1^2 - 4\alpha_1\beta_1$$
$$g_1 = 2 + \delta_1^2\eta^2, \qquad \sigma_{01} = \mu_0\mu_1^{-1}, \qquad g_0 = 2 + \eta^2$$

3. In the investigation of the displacement field in the half-space we follow [3] and deform the integration contour Re $\eta = \sigma$ into the stationary contours of the phase functions

$$f_1(\eta) = t\eta / b_0 - (z - h)\alpha_1, \qquad f_3(\eta) = t\eta / b_0 - (z - h)\beta_1 \tag{3.1}$$

In the case of displacements on the surface $z = 0$ we displace the contour Re $\eta = \sigma$ so as to include the branch cuts drawn from the branch points $\pm i\gamma_1^{-1}$ and $\pm i\delta_1^{-1}$.

With the above deformation the singular points of the integrands in the plane η may be intersected. In order to take singularities into consideration we have to investigate the integrands. Considering the explicit expressions of those functions we find that the integrands have an essential singularity $\eta = \infty$; branch points are $\pm i\gamma_1^{-1}$ and $\pm i\delta_1^{-1}$; singularities related to poles of $A(k, \eta)$ and poles coinciding with the roots of equation

$$\Delta_0 (kh, \eta) = 0 \tag{3.2}$$

In the investigation of the field represented by formulas (2.4) the behavior of the roots in relation to the parameter kh plays a decisive role. It is easy to see that for $kh = 0$ equation (3.2) has a finite number of roots located at a finite distance from the origin and an infinite number of roots at infinity in the left half-plane of the variable η. By analogy to [3] let us call the first group of roots the roots of the first class and the rest the roots of the second class. If we bear in mind that in the expansion into Fourier integral by the wave members, the frequency is determined by the expression $\omega = k$ Im ηb_0^{-1}, then it will turn out that the roots of the first class will correspond to oscillations whose spectrum begins with zero frequency. The roots of the second class correspond to oscillations which begin with the boundary frequencies

$$\omega_n = \lim_{k \to 0} (k \operatorname{Im} \eta b_0^{-1}) \approx n\pi (b_0 h)^{-1} \qquad (n = 1, 2, 3 \ldots)$$

Thus, for the investigation of the low-frequency oscillations it is sufficient to locate the roots of equation (3.2) for

$$kh \ll 1, \qquad kh\,|\eta| < 1 \tag{3.3}$$

These conditions are simultaneously fulfilled only for the roots of the first class.

A simple analysis of equation (3.2) shows that in the region (3.3) the integrands have two pairs of poles

$$\eta_1 = \pm i\tau_1 + O(k^2h^2), \qquad (\tau_1^2\delta_1^2 - 2)^2 - 4\sqrt{1-\gamma_1^2\tau_1^2}\sqrt{1-\delta_1^2\tau^2} = 0 \tag{3.4}$$

$$\eta_2 = \pm i2\sqrt{1-\gamma_0^2}\left[1 - \frac{k^2h^2}{6}(1-2\gamma_0^2)^2 \pm \pm \frac{2\sigma_{01}\delta_1^2(1-\gamma_0^2)^2(1-2\gamma_0^2)^2\alpha_1}{R_1\delta_0^2}k^3h^3\right] + O(k^4h^4) \tag{3.5}$$

and the poles related to the source function $A(k, \eta)$. (In (3.5) the values of α_1 and R_1 should be takne at the point $\eta = i2\sqrt{(1-\gamma_0^2)}$.) Since the present paper considers a low-frequency field due to an arbitrary source, the residues at the poles of the function $A(k, \eta)$ will not be especially investigated. The low-frequency field is thus reduced to the sum of integrals, with respect to k, of the residues at the poles (3.4) to (3.5) and of integrals along the stationary contours and branch cuts. The expressions containing integrals along the stationary contours describe maximum displacements on the surfaces

$$a_1^2\,[(z-h)^2 + r^2] = t^2, \qquad b_1^2\,\{(z-h)^2 - r^2] = t^2 \tag{3.6}$$

and therefore may be looked upon as low-frequency diffracted waves. The trace of those waves on the daylight surface is described by the integrals along the branch cuts. The residues at points η_1 determine a Rayleigh wave which is propagated in the layer and in the half-space along $z = h$. The properties of this wave are similar to those of a wave having the same name in a system consisting of the elastic half-space and a fluid layer [5-6].

As is well known, in a free plate under a symmetrical influence a longitudinal wave is propagated with the velocity $v = 2\sqrt{(1-\gamma_0^2 b_0^{-1})}$ (it is usually referred to as the longitudinal lamellar wave). It follows directly from (3.5) that the wave described by the residues at η_2 has the same propagation velocity. However, for the model of the medium considered here the character of dispersion of the wave, as well as its damping, are different. Under the condition $b_0 < 2b_1\sqrt{(1-\gamma_0^2)}$ this wave undergoes additional exponential damping as it is propagated. The magnitude of this damping depends on Re η_2 which is expressed as

$$\operatorname{Re}\eta_2 = -\varkappa_1(kh)^3 = -\left.\frac{4\sigma_{01}\delta_1^2(1-\gamma_0^2)^{5/2}(1-2\gamma_0^2)^2|\alpha_1|\,k^3h^3}{g_1^2 + 4\,|\alpha_1|\,|\beta_1|}\right|_{\eta=i2\sqrt{1-\gamma_0^2}}$$

$$2\sqrt{1-\gamma_0^2} > \gamma_1^{-1} \tag{3.7}$$

$$\operatorname{Re}\eta_2 = -\varkappa_2 (kh)^3 = \frac{-16\sigma_{01}\delta_1^2(1-\gamma_0^2)^{5/2}(1-2\gamma_0^2)^2|\alpha_1|^2|\beta_1|k^3h^3}{g_1^4+16|\beta_1^2||\alpha_1^2|}\Bigg|_{\eta=i2\sqrt{1-\gamma_0^2}}$$

$$\delta_1^2 < 2\sqrt{1-\gamma_0^2} < \gamma_1^{-1}$$

4. Below we will distinguish three cases of relations of parameters

$$a_0^{-1} > b_0^{-1} > a_1^{-1} > b_1^{-1} \tag{4.1}$$
$$a_1^{-1} > a_0^{-1} > b_1^{-1} > b_0^{-1} \tag{4.2}$$
$$a_1^{-1} > b_1^{-1} > a_0^{-1} > b_0^{-1} \tag{4.3}$$

Substituting (3.5) and (3.7) into (2.3) and changing the variables $\omega = 2k\sqrt{(1-\gamma_0^2 b_0^{-1})}$ we can obtain the spectral representation of the field for each of the cases (4.1) to (4.3) in the following general final form:

$$q_0 = \operatorname{Re}\int_0^{\omega_1} A_0(\omega) J_1\left(\frac{r\omega b_0}{2\sqrt{1-\gamma_0^2}}\right)\exp\left[-\chi_\nu\left(\frac{h}{\lambda}\right)^3\right]e^{i\omega t}\,d\omega$$

$$w_0 = \operatorname{Re}\int_0^{\omega_1} B_0(\omega) J_0\left(\frac{r\omega b_0}{2\sqrt{1-\gamma_0^2}}\right)\exp\left[-\chi_\nu\left(\frac{h}{\lambda}\right)^3\right]\left(\frac{h}{\lambda}\right)e^{i\omega t}\,d\omega \tag{4.4}$$

$$q_1 = \operatorname{Re}\int_0^{\omega_1} J_1\left(\frac{r\omega b_0}{2\sqrt{1-\gamma_0^2}}\right)\left\{C_1^{(1)}(\omega)\exp\left[i\omega\left(t-\frac{b_0(z-h)}{2\sqrt{1-\gamma_0^2}}\sqrt{4\gamma_1^2(1-\gamma_0^2)-1}\right)\right]+\right.$$
$$\left.+C_1^{(2)}(\omega)\exp\left[i\omega\left(t-\frac{b_0(z-h)}{2\sqrt{1-\gamma_0^2}}\sqrt{4\delta_1^2(1-\gamma_0^2)-1}\right)\right]\right\}\left(\frac{h}{\lambda}\right)^2\exp\left[-\chi_\nu\left(\frac{h}{\lambda}\right)^3\right]d\omega$$

$$w_1 = \operatorname{Re}\int_0^{\omega_1} J_0\left(\frac{r\omega b_0}{2\sqrt{1-\gamma_0^2}}\right)\left[D_1^{(1)}(\omega)\exp\left[i\omega\left(t-\frac{b_0(z-h)}{2\sqrt{1-\gamma_0^2}}\sqrt{4\gamma_1^2(1-\gamma_0^2)-1}\right)\right]+\right.$$
$$\left.+D_1^{(2)}(\omega)\exp\left[i\omega\left(t-\frac{b_0(z-h)}{2\sqrt{1-\gamma_0^2}}\sqrt{4\delta_1^2(1-\gamma_0^2)-1}\right)\right]\right\}\left(\frac{h}{\lambda}\right)^2\exp\left[-\chi_\nu\left(\frac{h}{\lambda}\right)^3\right]d\omega \tag{4.5}$$

where

$$\chi_\nu = \varkappa_\nu\left(\frac{\pi\gamma_0}{\sqrt{1-\gamma_0^2}}\right)^3\frac{\omega t}{2\sqrt{1-\gamma_0^2}} \tag{4.6}$$

and $\kappa_\nu = \kappa_1$ if (4.1) is fulfilled, $\kappa_\nu = \kappa_2$ in the case of (4.2) and, finally, $\kappa_\nu = 0$, if (4.3) takes place. The explicit form of the functions C, A and B, D, introduced into (4.4) to (4.5), can be easily determined from (2.3) and (2.4), and h/λ is equal to the relation of the plate thickness to the wave length in the plate $\lambda = (a_0 f)^{-1}$. In accordance with (3.3) the frequency ω_1, which is the upper limit of integration, satisfies the condition

$$\omega_1 \ll 2\sqrt{1-\gamma_0^2}\,(b_0h)^{-1} \tag{4.7}$$

In the discussion of the physical results which follow from the solution of the problem at hand, attention is called to the fact that the bending oscillations, which exist in the case of a free layer, are absent here. The lamellar longitudinal wave has the same dispersion as in the free layer. However, the damping of the wave with the distance by the cylindrical law takes place only if the inequality (4.3) is fulfilled. In that case there is no loss of energy into the underlying half-space, in which the displacement field is a nonhomogeneous wave (diminishes exponentially with the increase of z). If conditions (4.1) and (4.2) are fulfilled, the lamellar wave in the layer undergoes additional damping due to the radiation of energy into the half-space. The amplitude q_0 of the component of displacement vector diminishes monotonously as h/λ increases, and the amplitude w_0 of the component has a maximum whose location is determined by the equality

$$\frac{h}{\lambda} = \sqrt[3]{\frac{1}{3\chi_\nu}} \tag{4.8}$$

from which it follows that, as t increases, the maximum is displaced towards the smaller values of h/λ; in that case homogeneous conical waves will be excited in the half-space. In the case (4.1) both the longitudinal and the transverse waves will be homogeneous, in the case (4.2) the transverse wave will be homogeneous, and the longitudinal will be nonhomogeneous. The equation of surfaces of equal phase is easily obtained by using the asymptotic expansion of Bessel functions for large r

$$\begin{aligned} t - \frac{b_0 r}{2\sqrt{1-\gamma_0^2}} - \frac{b_0(z-h)}{2\sqrt{1-\gamma_0^2}}\sqrt{4\gamma_1^2(1-\gamma_0^2)-1} &= \text{const} \\ t - \frac{b_0 r}{2\sqrt{1-\gamma_0^2}} - \frac{b_0(z-h)}{2\sqrt{1-\gamma_0^2}}\sqrt{4\delta_1^2(1-\gamma_0^2)-1} &= \text{const} \end{aligned} \tag{4.9}$$

The location of the maximum of spectral function of these oscillations is determined by the condition

$$\frac{h}{\lambda} = \sqrt[3]{\frac{2}{3\chi_\nu}}$$

which characterizes the displacement of the maximum towards smaller values of h/λ as r increases. It is interesting that as the wave propagates the amount of damping is higher for higher values of h than for smaller ones. In conclusion, it should be noted that if the boundary between the layer and the half-space is fixed, the low-frequency waves of the type considered here do not arise [3].

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